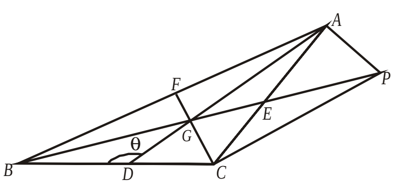
Indian National Mathematical Olympiad & Regional Mathematical Olympiad Problems

1. Let G be the centroid of the triangle ABC in which the angle at C is obtuse and let AD and CF be medians from A and C respectively onto the sides BC and AB. If the four points B, D, G and F are concyclic, show that $\frac{AC}{BC} > \sqrt{2}$. If, further, P is a point on the line BG extended such that $AGCP$ is a parallelogram, show that the triangles ABC and GAP are similar.

Soln: If $\angle ADB = \theta$ then from triangles ABD and ADC we get

$$AB^2 = AD^2 + BD^2 - 2AD \,.\, BD \,.\, \cos\theta$$
$$AC^2 = AD^2 + DC^2 + 2AD \,.\, DC \cos\theta$$

Adding,



$$c^2 + b^2 = 2AD^2 + \frac{1}{2}a^2$$

i.e. $AD^2 = \frac{b^2 + c^2}{2} - \frac{a^2}{4}$.

Similarly, the other medians are given by

$$BE^2 = \frac{c^2 + a^2}{2} - \frac{b^2}{4}, CF^2 = \frac{a^2 + b^2}{2} - \frac{c^2}{4}$$

Since B, D, G, F lie on a circle we have

$$AF \,.\, AB = AG \,.\, AD$$

But $AG = \frac{2}{3}AD$.

Hence $\frac{1}{2}c^2 = \frac{2}{3}AD^2 = \frac{1}{3}(b^2 + c^2 - \frac{a^2}{2})$

Þ $\frac{3}{2}c^2 = b^2 + c^2 - \frac{1}{2}a^2 \Rightarrow b^2 = \frac{1}{2}(c^2 + a^2)$

For the first part we have

$$b^2 = c^2 + a^2 - 2ca \cos B = 2b^2$$
$$- 2ca \cos B;$$

i.e.
$$b^2 = 2ca \cos B,$$
$$a = c \cos B + b \cos C < c \cos B$$

since $C > 90°$. Hence $2a^2 < 2c \cos B = b^2$; $\frac{b}{a} > \sqrt{2}$.

This proves the first part.

For the second part, let the line passing through C and parallel to AG meet BG produced in P. Given that $AGCP$ is a parallelogram. So AC and GP have the same midpoint E. Hence

$$GP = 2GE = \frac{2}{3} BE$$

$$AG = \frac{2}{3}AD, AP = CG = \frac{2}{3}CF$$

Since

$$b^2 = \frac{1}{2}(c^2 + a^2) \text{ we get}$$

$$AD^2 = \frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{4}a^2 = \frac{3}{4}c^2$$

$$BE^2 = \frac{1}{2}c^2 + \frac{1}{2}a^2 - \frac{1}{4}b^2 = \frac{3}{4}b^2$$

$$CF^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{4}c^2 = \frac{3}{4}a^2$$

Hence

$$AG = \frac{2}{3} \cdot \frac{\sqrt{3}}{2} c = \frac{1}{\sqrt{3}} c = \frac{1}{\sqrt{3}} c, GP = \frac{2}{3} - \frac{\sqrt{3}}{2} b = \frac{1}{\sqrt{3}} b,$$

$$PA = \frac{2}{3} . \frac{\sqrt{3}}{2} a = \frac{1}{\sqrt{3}} a.$$

Thus $$\frac{AG}{BA} = \frac{GP}{AC} = \frac{PA}{CB} = \frac{1}{\sqrt{3}}.$$

So AGP is similar to ABC.

2. If $x^5 - x^3 + x = a$ prove that $x^6 \geq 2a - 1$

Soln: Given that $a = x(x^4 - x^2 + 1) = x(x^6 + 1) / (x^2 + 1)$. If $x \leq 0$ then this shows that $a \leq 0$ and hence $2a - 1 < 0 \leq x^6$. Suppose that $x > 0$, then $a > 0$ and

$$x^6 + 1 = a\left(\frac{x^2+1}{x}\right) = a\left(x + \frac{1}{x}\right) \geq 2a;$$

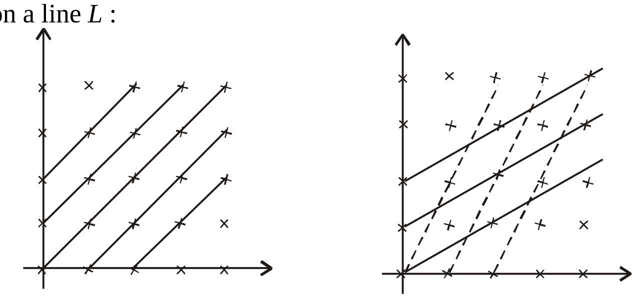
hence $x^6 \geq 2a - 1$.

3. In any set of 181 square integers, prove that one can always find a subset of 19 numbers, sum of whose elements is divisible by 19.

Soln: Consider the remainders obtained when the given 181 square integers are divided by 19. If n is any integer then n is congruent modulo 19 to one of the numbers 0, 1, 2, . . ., 18 and hence n^2 is congruent to 0, 1, 4, 9, 16, 6, 17, 11, 7 or 5. Hence there are only 10 possible values for the remainders. Since there are 181 remainders at least one of them must repeat 19 times (or) more. Choose 19 numbers congruent to that remainder. This proves the result.

4. Find the number of nondegenerate triangles whose vertices lie in the set of points (s, t) in the plane such that $0 \leq s \leq 4$, $0 \leq t \leq 4$, s and t are integers.

Soln: There are 25 points in the given set. We can choose 3 out of them in $\binom{25}{3}$ ways. Let us count the number of ways in which the 3 points chosen will lie on a line L :



(i) (ii)

The given set S contains 5 horizontal lines having 5 points each. We can choose 3 points from any of them in $\binom{5}{3}$ ways. Hence the number of ways in which L can be a horizontal line is $5.\binom{5}{3} = 50$. Similarly the number of ways in which L can be a vertical line is 50.

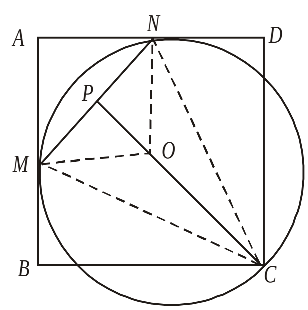
As shown in fig. (i) S contains 5 lines of slope 1; one line contains 5 points, 2 lines contain 4 points each and 2 lines contain 3 points each. So the number of ways in which L can be line of slope 1 is $\binom{5}{3} + 2\binom{4}{3} + 2\binom{3}{3} = 20$.

Similarly, the number of ways in which L can be a line of slope –1 is 20.

As shown in fig (ii) there are 3 lines of slope $\frac{1}{2}$ each containing 3 points; and there are 3 lines of slope 2, each containing 3 points. So the number of ways in which L can have slope $\frac{1}{2}$ or 2 is $6\binom{3}{3} = 6$ Similarly L can have slope $-\frac{1}{2}$ or – 2 in 6 ways.

Since no other line can contain more than two points of S, the number of ways in which the 3 points chosen will lie on a line is 50 + 50 + 20 + 20 + 6 + 6 + = 152. The required number of triangles is therefore, $\binom{25}{3} - 152 = 2148$.

5. A circle passes through the vertex C of a rectangle $ABCD$ and touches its sides AB and AD at M and N respectively. If the distance from C to the line segment MN is equal to 5 units find the area of the rectangle $ABCD$.



Soln: Let O be the centre of the circle and P be the foot of perpendicular from C to MN. Then OM is perpendicular to AB, ON is perpendicular to AD and $OM = ON$ = the radius of the circle. So $AMON$ is a square.

$$\text{Đ}\, MCN = \frac{1}{2}\,\text{Đ}\, MON = 45°$$

$$\text{Đ}\, CMP + \text{Đ}\, CNP = 135° = \text{Đ}\, CMP + \text{Đ}\, CMB = \text{Đ}\, CNP + \text{Đ}\, CND.$$

Hence Đ CNP = Đ CMB and Đ CMP = Đ CND. Thus we see that the right triangles CNP and CMB are similar, and CMP and CND are similar. So

$$\frac{CN}{CM} = \frac{CP}{CB}, \quad \frac{CM}{CN} = \frac{CP}{CD}$$

Multiplying, $1 = \dfrac{CP^2}{CB\,.\,CD}$. Hence the area of the rectangle is

$$CB\,.\,CD = CP^2 = 5^2 = 25.$$

Alternately : Let $AB = a$, $BC = b$, $BM = x$, $DN = y$ and r = radius of the circle. Producing MO to meet the opposite side we can see that $x^2 + y^2 = OC^2 = r^2$. Thus

$$\begin{aligned} CM^2 &= b^2 + x^2 = (r + y)^2 + x^2 \\ &= 2r^2 + 2ry = 2br; \\ CN^2 &= a^2 + y^2 = (r + x)^2 + y^2 \\ &= 2r^2 + 2rx = 2ar \end{aligned}$$

Also $\dfrac{CP}{CN} = \sin \angle CNP$; $CM = 2r \sin \angle CNM$.

Hence $CM.\, CN = 2r\,.\,CP = 10r$. Thus $(2ar)(2br) = (10r)^2 \Rightarrow ab = 25$.

6. If $f : \mathrm{R} \to \mathrm{R}$ is a function satisfying the properties

(i) $f(-x) = -f(x)$

(ii) $f(x + 1) = f(x) + 1$

(iii) $f\left(\dfrac{1}{x}\right) = \dfrac{f(x)}{x^2}$ for $x \neq 0$

prove that $f(x) = x$ for all real values of x. (Here R denotes the set of all real numbers).

Soln: Let $x \neq 0$, $x \neq -1$. Then

$$f\left(\frac{1}{x} + 1\right) = f\left(\frac{1}{x}\right) + 1;$$

i.e.

$$f\left(\frac{1 + x}{x}\right) = \frac{f(x)}{x^2} + 1$$

Hence,

$$\begin{aligned} f\left(\frac{x}{x+1}\right) &= f\left(\frac{x+1}{x}\right) \cdot \left(\frac{x+1}{x}\right)^2 \\ &= \left(\frac{x}{x+1}\right)^2 \left\{\frac{f(x)}{x^2} + 1\right\} \end{aligned}$$

$$(x+1)^2 f\left(\frac{x}{x+1}\right) = f(x) + x^2$$

But $$\frac{x}{x+1} = 1 - \frac{1}{x+1}.$$

So $$f\left(\frac{x}{x+1}\right) = f\left(-\frac{1}{x+1}\right) + 1$$

$$= -f\left(\frac{1}{x+1}\right) + 1$$

$$= -\frac{f(x+1)}{(x+1)^2} + 1;$$

$$(x+1)^2 f\left(\frac{x}{x+1}\right) = -f(x+1) + (x+1)^2$$

$$= -f(x) - 1 + x^2 + 2x + 1$$

Hence $$f(x) + x^2 = (x+1)^2 f\left(\frac{x}{x+1}\right)$$

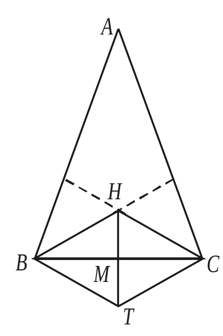
$$= -f(x) + x^2 + 2x;$$

$$2f(x) = 2x \Rightarrow f(x) = x.$$

Taking $x = 0$ in (i) we get $f(0) = 0$; so (ii) gives $0 = f(-1+1) = f(-1) + 1 \Rightarrow f(-1) = -1$. Hence $f(x) = x$ for all x in R.

7. In an acute-angled triangle ABC, $\angle A = 30°$, H is the orthocentre, and M is the mid point of BC. On the line HM, take a point T such that $HM = MT$. Show that $AT = 2BC$.

Soln: The diagonols BC and TH of the quadrilateral $BTCH$ are bisected at M. Hence $BTCH$ is a parallelogram. Since $CH \perp AB$ and $CH \parallel TB$, we have $TB \perp AB$. Similarly $TC \perp AC$ since $BH \perp AC$. Thus the circle on AT as diameter passes through B and C. This is the circumcircle of triangle ABC. If R is its radius then



$AT = 2R$, $BC = 2R \sin A = 2R \sin 30° = R$

Hence

$$AT = 2BC.$$

Alternately. We can assume that the circumcentre of ΔABC is at the origin. If R is the circum radius, $BC = 2R \sin A = R$. Also if Z_1, Z_2, Z_3 are complex numbers representing A, B, C respectively than $Z_1 + Z_2 + Z_3$ represents H and M is $\frac{Z_2 + Z_3}{2}$. If t respresents T,

then $\frac{t + Z_1 + Z_2 + Z_3}{2} = \frac{Z_2 + Z_3}{2} \Rightarrow t = -Z_1 \Rightarrow AT = 2R = 2BC.$

8. Show that there are infinitely many pairs (a, b) of relatively prime integers (not necessarily positive) such that both quadratic equations

$$x^2 + ax + b = 0$$

and

$$x^2 + 2ax + b = 0$$

have integer roots.

Soln: Since b is the product of roots of the quadratics, it must be expressible as a product of integers in two different ways. So let us try for b numbers of the form uvw. Suppose that

$$u + vw = a, \quad uv + w = 2a$$

Then $2u + 2vw = uv + w$; $w(2v - 1) = u(v - 2)$. This will be satisfied if $u = 2v - 1$, $w = v - 2$ where v is any integer. If $v = n$ this gives

$$a = (2n - 1) + n(n - 2) = n^2 - 1,$$
$$b = n(n - 2)(2n - 1) \qquad ...(1)$$

If a and b are chosen like this, then the roots of the first equation are

$$-u = -(2n - 1), -vw = -n(n - 2);$$

those of the second equation are

$$-uv = -n(2n - 1), -w = -(n - 2).$$

These are integers. So if a and b are relatively prime then the conditions are fulfilled. Let p be any prime number dividing $a = (n - 1)(n + 1)$. If p divides $n - 1$, then p does not divide any of the numbers:

$$n - 1 + 1 = n, \quad n - 1 - 1 = n - 2, \quad 2(n - 1) + 1 = 2n - 1;$$

hence p does not divide b. Suppose that p divides $n + 1$. Then p does not divide

$$n + 1 - 1 = n, \quad n + 1 - 3 = n - 2, \quad 2(n + 1) - 3 = 2n - 1,$$

provided $p \neq 3$. If $n + 1$ is not divisible by 3 this requirement also is satisfied. So a and b cannot have a common prime divisor if n is any integer of the form $3k$ or $3k + 1$, where k is any integer. Thus there are infinitely many pairs (a, b), given by (1), where $n + 1$ is not divisible by 3, satisfying the given conditions.

If $n + 1$ is divisible by 3, and a, b are given by (1) then a is divisible by 3 and b is divisible by 9. It can be seen that $(a/3, b/9)$ will satisfy our conditions. Clearly, if (a, b) satisfies the conditions, then $(-a, b)$ also will satisfy the conditions.

9. Show that the number of 3 element subsets a, b, c of 1, 2, 3, . . ., 63 with $a + b + c < 95$ is less than the number of those with $a + b + c > 95$.

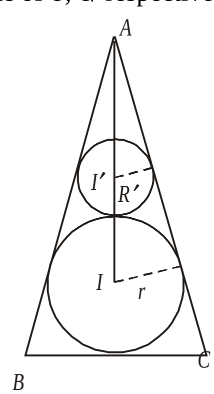
Soln: Consider the sums $a + b + c$ of all 3 element subsets a, b, c of $S = \{1, 2, 3, \ldots 63\}$. Divide the class of all 3 element subsets of S into four classes I, II, III and IV according as their sums are less than 95, greater than 97, equal to 96 or 97 and equal to 95 respectively. To each $A = \{a, b, c\}$ of class I associate a subset $A_1 = \{a_1, b_1, c_1\}$ where $a_1 = 64 - a$, $b_1 = 64 - b$, $c_1 = 64 - c$. Then $a_1 + b_1 + c_1 = 192 - (a + b + c) > 97$; hence A_1 belong to class II. It is clear that distinct sets in class I are associated with distinct sets in class II. So class II contains as many members as class I. Class II is not empty since {31, 32, 33) belongs to it. Hence the classes II and III together contain more subsets than class I. This completes the proof.

10. Let ABC be a triangle and a circle Γ' be drawn lying inside the triangle, touching its incircle Γ externally and also touching the two sides AB and AC. Show that the ratio of the radii of the circles Γ' and Γ is equal to $\tan^2\left(\frac{\pi - A}{4}\right)$

Soln: Let r, r' be the radii, and I, I' be the centres of Γ, Γ' respectively. Then

$$\frac{r}{AI} = \sin\frac{A}{2} = \frac{r'}{AI'}$$

Hence



$$\sin\frac{A}{2} = \frac{r - r'}{AI - AI'}$$

$$= \frac{r - r'}{II'};$$

$$\frac{\sin A/2}{1} = \frac{r - r'}{r + r'};$$

$$\frac{1 - \sin A/2}{1 + \sin A/2} = \frac{(r + r') - (r - r')}{(r + r') + (r - r')} = \frac{2r'}{2r}$$

$$\frac{r'}{r} = \frac{1 - \cos(\pi/2 - A/2)}{1 + \cos(\pi/2 - A/2)}$$

$$= \frac{2\sin^2\left(\frac{\pi - A}{4}\right)}{2\cos^2\left(\frac{\pi - A}{4}\right)}$$

$$= \tan^2\left(\frac{\pi - A}{4}\right)$$

This proves the result.